

# Multiple expansions for energy and momenta carried by gravitational waves

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**Abstract** We present expressions for the energy, linear momentum and angular momentum carried away from an isolated system by gravitational radiation based on spin-weighted spherical harmonics decomposition of the Weyl scalar  $\Psi_4$ . We also show that the expressions derived are equivalent to the common expressions obtained when using a framework based on perturbations of a Schwarzschild background. The main idea is to collect together all the different expressions in a uniform and consistent way. The formulae presented here are directly applicable to the calculation of the radiated energy, linear momentum and angular momentum starting from the gravitational waveforms which are typically extracted from numerical simulations.

**Keywords** Gravitational waves · Energy and momenta · Multipole expansions · Numerical relativity

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## 1 Introduction

Gravitational waves are one of the most important predictions of General Relativity. Though such gravitational radiation has not yet been detected directly, there is strong indirect evidence for its existence in the form of the now famous binary pulsar PSR 1913+16, whose change in orbital period over time matches to very high accuracy the value predicted by General Relativity as a consequence of the emission of gravitational waves [1,2]. Moreover, there is every reason to believe that the new generation of large interferometric observatories (LIGO, VIRGO, GEO 600, TAMA) will finally succeed in detecting gravitational radiation within the next few years. Also, gravitational waves are one of the most important physical phenomena associated with the presence of strong and dynamic gravitational fields, and as such they are of great interest in numerical relativity.

Gravitational radiation can carry energy and momentum away from an isolated system, and it also encodes important information about the physical properties of the system itself. The prediction of the gravitational wave signal coming from the inspiral collision of two compact objects has been one of the main themes in numerical relativity over the years, as such predictions can be used as templates that can significantly improve the possibility of detection. Nevertheless, it was not until a couple of years ago that it was finally possible to perform long-term stable numerical simulations of binary black hole spacetimes [3–7]. In this way, numerical simulations have now reached a stage where it is finally possible to extract important astrophysical information from the collision of two black holes. For instance, it has now become possible to compute the so-called “kick” of the final black hole, i.e., the non-zero final velocity of the merged black hole, see, e.g., [8,9] and references therein (we have not attempted to give a complete list of references here since this field is so active at the moment that such an attempt would almost certainly become obsolete within a few weeks). This non-zero final velocity is a consequence of the fact that in non-symmetric situations the emitted gravitational waves carry linear momentum with them. Similarly, the spin of the final black hole can also be computed by subtracting the angular momentum carried away by gravitational waves from the initial ADM angular momentum of the spacetime.

Historically, there have been two main approaches to the extraction of gravitational wave information from a numerical simulation. For a number of years the traditional approach has been based on the theory of perturbations of a Schwarzschild spacetime, developed originally by Regge and Wheeler [10], Zerilli [11], and a number of other authors, and later recast as a gauge invariant framework by Moncrief [12]. In recent years, however, it has become increasingly common in numerical relativity to describe gravitational waves in terms of the components of the Weyl curvature tensor with respect to a frame of null vectors [13], using what is known as the Newman–Penrose formalism [14]. In this paper we obtain in a simple way general expressions for the energy, linear momentum and angular momentum carried away by gravitational waves using the spin-weighted spherical harmonics decomposition of the components of the Weyl curvature tensor. We also compare these expressions with the more common expressions used in numerical relativity which are based on the theory of gauge invariant perturbations of Schwarzschild [15] (see, e.g. [9,16]). Though most of the

main ideas and results presented here are known, our aim is to collect all relevant expressions together using a consistent set of conventions and definitions.

This paper is organized as follows. In Sect. 2 we summarize the general expression for the energy and momenta of gravitational waves in the transverse-traceless (TT) gauge. In Sect. 3 we obtain the general expressions for the energy and momenta using the Weyl scalars and the spin-weighted spherical harmonics. Later, in Sect. 4 we briefly discuss some of the ideas behind the theory of perturbations of a Schwarzschild black hole and we obtain again the general expression for energy and momenta in terms of the gauge invariant “master” functions ( $\Psi_{\text{even}}, \Psi_{\text{odd}}$ ) and ( $Q_{\text{even}}, Q_{\text{odd}}$ ). We conclude in Sect. 5. In addition, in Appendix A we discuss some important properties of the spin-weighted spherical harmonics.

## 2 Gravitational waves and the physics in them encoded

Let us start by considering a small perturbation  $h_{\mu\nu}$  to a flat background metric  $\hat{g}_{\mu\nu}$ , such that the full metric is given by

$$g_{\mu\nu} = \hat{g}_{\mu\nu} + h_{\mu\nu}, \tag{1}$$

where  $|h_{\mu\nu}| \ll 1$ . Knowing that the gravitational field has only two degrees of freedom, we can choose a gauge such that there are only two unknown functions,  $h^+$  and  $h^\times$ , that represent the two possible polarizations of the gravitational waves. That is, we work in the so-called *transverse-traceless* (TT) gauge in which imposes the conditions  $h_{0\alpha} = h^i{}_i = h^{ij}{}_{|j} = 0$  (where the bar denotes covariant derivative with respect to the background metric). The perturbation tensor then has the form

$$h^{TT}{}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h^+ & h^\times & 0 \\ 0 & h^\times & -h^+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2}$$

where we have assumed that we have a plane wave propagating along the  $z$  axis. Moreover, we will be considering only outgoing waves which means that the functional dependence on  $r$  and  $t$  of the metric perturbations is of the form  $f(r - t)/r$ , so that asymptotically one has  $\partial_r h \sim -\partial_t h \equiv -\dot{h}$ . Furthermore, for large  $r$  the radiation can always be locally approximated as a plane wave, so that the angular derivatives can be neglected when compared with radial derivatives (but one must be careful when one deals with quantities that do not involve radial derivatives such as the angular momentum, see below).

Having described the properties of the metric perturbation  $h_{\mu\nu}$ , we proceed to find the flux of energy and momentum carried away by the gravitational waves. The most straightforward way is to use the Isaacson stress-energy tensor (for details see, e.g. [17]), which describes the energy and momentum associated with the gravitational waves averaged over a few wavelengths using the so-called short wavelength approximation (one must remember that in general relativity there is no local expression for

the energy and momentum of the gravitational field). In the TT gauge, and in a locally inertial frame, the Isaacson stress-energy tensor is given by

$$T_{\mu\nu} = \frac{1}{32\pi} \left\langle \partial_\mu h_{ij}^{TT} \partial_\nu h_{ij}^{TT} \right\rangle, \tag{3}$$

where  $\langle \rangle$  denotes an average over several wavelengths, and where a summation over the repeated indices  $(i, j)$  is implied. Using the explicit form of the  $h_{ij}^{TT}$  in terms of  $h_+$  and  $h_\times$ , the Isaacson stress-energy tensor can be rewritten as

$$T_{\mu\nu} = \frac{1}{16\pi} \left\langle \partial_\mu h^+ \partial_\nu h^+ + \partial_\mu h^\times \partial_\nu h^\times \right\rangle, \tag{4}$$

or equivalently (here and in what follows  $\bar{z}$  will denote the complex conjugate of  $z$ )

$$T_{\mu\nu} = \frac{1}{16\pi} \operatorname{Re} \left\langle \partial_\mu H \partial_\nu \bar{H} \right\rangle, \tag{5}$$

with  $H := h^+ - ih^\times$ , and where  $\operatorname{Re}(z)$  denotes the real part of  $z$ .

We can now use Eq. (5) to find the flux of energy and momentum carried away by the gravitational waves. Consider first the flux of energy along the direction  $i$ , which is given in general by  $T^{0i}$ . In particular, the energy flux of the gravitational waves along the radial direction will then be given in local Cartesian coordinates by

$$\frac{dE}{dt dA} = T^{0r} = \frac{1}{16\pi} \operatorname{Re} \left\langle \partial^0 H \partial^r \bar{H} \right\rangle = -\frac{1}{16\pi} \operatorname{Re} \left\langle \partial_t H \partial_r \bar{H} \right\rangle, \tag{6}$$

with  $dA$  the area element normal to the radial direction. Using now the relation  $\partial_r h = -\dot{h}$  mentioned above, we can rewrite this as

$$\frac{dE}{dt dA} = \frac{1}{16\pi} \left\langle \dot{H} \dot{\bar{H}} \right\rangle = \frac{1}{16\pi} \left\langle |\dot{H}|^2 \right\rangle. \tag{7}$$

If we want the total flux of energy leaving the system at a given time we need to integrate over the sphere to find

$$\frac{dE}{dt} = \lim_{r \rightarrow \infty} \frac{r^2}{16\pi} \oint |\dot{H}|^2 d\Omega, \tag{8}$$

where we have taken  $dA = r^2 d\Omega$ , with  $d\Omega$  the standard solid angle element, and where the limit of infinite radius has been introduced since the Isaacson stress-energy tensor is only valid in the weak field approximation. Notice also that we have dropped the averaging since the integral over the sphere is already performing an average over space, plus the expression above is usually integrated over time to find the total energy radiated which again eliminates the need to take a time average.

Consider next the flux of linear momentum which corresponds to the spatial components of the stress-energy tensor  $T^{ij}$ . The flux of momentum  $i$  along the radial

direction will then be given by

$$\frac{dP_i}{dt dA} = T_{ir} = \frac{1}{16\pi} \operatorname{Re} \langle \partial_i H \partial_r \bar{H} \rangle = \frac{1}{16\pi} l_i \langle |\dot{H}|^2 \rangle, \tag{9}$$

where in the last inequality we have used the fact that asymptotically  $\partial_i H \simeq (x_i/r) \partial_r H$  (i.e., we are ignoring angular derivatives), and also the relation between the radial and temporal derivatives for outgoing waves. The vector  $\mathbf{l}$  introduced above is the unit radial vector in flat space,

$$\mathbf{l} = \mathbf{x}/r = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta). \tag{10}$$

The total flux of momentum leaving the system will again be given by an integral over the sphere as

$$\frac{dP_i}{dt} = \lim_{r \rightarrow \infty} \frac{r^2}{16\pi} \oint l_i |\dot{H}|^2 d\Omega. \tag{11}$$

Finally, we consider the flux of angular momentum. Locally, the flux of the  $i$  component of the angular momentum along the radial direction should correspond to  $\epsilon_{ijk} x^j T^{kr}$  with  $\epsilon_{ijk}$  the three-dimensional Levi–Civita antisymmetric tensor (this is just  $\mathbf{r} \times \mathbf{p}$  in three-dimensional notation). However, in the case of gravitational waves, this expression is in fact wrong since the averaging procedure that is used to derive the Isaacson stress-energy tensor ignores terms that go as  $1/r^3$ , and it is precisely such terms that contribute to the flux of angular momentum. A correct expression for the flux of angular momentum due to gravitational waves was first derived by DeWitt in 1971, and in the TT gauge has the form (see, e.g. [15])

$$\frac{dJ^i}{dt dA} = \frac{1}{32\pi} \epsilon^{ijk} (x_j \partial_k h_{ab} + 2 \delta_{aj} h_{bk}) \partial_r h^{ab}. \tag{12}$$

The last expression can be rewritten in more compact form if we introduce the angular vectors  $\xi_i$  associated to rotations around the three coordinate axis. These vectors are Killing fields of the flat metric, and in Cartesian coordinates have components given by  $\xi_i^k = \epsilon_i{}^{jk} x_j$  (where  $\xi_i^k$  represents the  $k$  component of the vector  $\xi_i$ ). In terms of the vectors  $\xi_i$  the flux of angular momentum can now be written as

$$\frac{dJ_i}{dt} = - \lim_{r \rightarrow \infty} \frac{r^2}{32\pi} \oint (\mathfrak{L}_{\xi_i} h_{ab}) \partial_r h^{ab} d\Omega, \tag{13}$$

where  $\mathfrak{L}_{\xi_i} h_{ab}$  is the Lie derivative of  $h_{ab}$  with respect to  $\xi_i$ , and where we have again taken  $\partial_r h = -\partial_t h$  for outgoing waves. The appearance of the Lie derivative is to be expected on physical grounds, since for rotational symmetry around a given axis ( $\mathfrak{L}_{\xi_i} h_{ab} = 0$ ), one should find that the corresponding angular momentum flux vanishes.

In order to write the angular momentum flux in terms of  $H$  as we have done with the flux of energy and linear momentum, one must now carefully consider the action of the Lie derivative on the perturbed metric. The easiest way to do this is to work in

spherical coordinates  $(r, \theta, \varphi)$ , in which case the angular vectors have components

$$\xi_x = (0, -\sin \varphi, -\cos \varphi \cot \theta), \tag{14}$$

$$\xi_y = (0, \cos \varphi, -\sin \varphi \cot \theta), \tag{15}$$

$$\xi_z = (0, 0, 1). \tag{16}$$

It is clear that the vector  $\xi_z$  corresponds to one of the vectors of the coordinate basis, which implies that Lie derivatives along it reduce to simple partial derivatives. To calculate the Lie derivatives in the other directions it is convenient to first introduce the two complex angular vectors  $\xi_{\pm} := \xi_x \pm i \xi_y$ . Furthermore, we will introduce an orthonormal spherical basis  $(\hat{e}_r, \hat{e}_\theta, \hat{e}_\varphi)$ , and define the two unit complex vectors  $\hat{e}_{\pm} := 1/\sqrt{2} (\hat{e}_\theta \mp i \hat{e}_\varphi)$ . One can then show with some algebraic equations that the Lie derivative of  $\hat{e}_{\pm}$  with respect to  $\xi_{\pm}$  is given by

$$\mathfrak{L}_{\xi_{\pm}} \hat{e}_{\pm}^a = \mp \left( i e^{\pm i \varphi} \csc \theta \right) e_{\pm}^a. \tag{17}$$

Let us now rewrite the metric perturbation  $h_{ab}$  in the TT gauge in terms of the orthonormal basis as:

$$\begin{aligned} h_{ab} &= h^+ [(\hat{e}_\theta)_a (\hat{e}_\theta)_b - (\hat{e}_\varphi)_a (\hat{e}_\varphi)_b] \\ &\quad + h^\times [(\hat{e}_\theta)_a (\hat{e}_\varphi)_b + (\hat{e}_\varphi)_a (\hat{e}_\theta)_b] \\ &= H (\hat{e}_-)_a (\hat{e}_-)_b + \bar{H} (\hat{e}_+)_a (\hat{e}_+)_b. \end{aligned} \tag{18}$$

We are now in a position to calculate the Lie derivative of  $h_{ab}$  with respect to  $\xi_{\pm}$ . One finds,

$$\mathfrak{L}_{\xi_{\pm}} h_{ab} = (\hat{e}_-)_a (\hat{e}_-)_b \hat{J}_{\pm} H + (\hat{e}_+)_a (\hat{e}_+)_b \hat{J}_{\pm} \bar{H}, \tag{19}$$

where we have defined the operators

$$\begin{aligned} \hat{J}_{\pm} &:= \xi_{\pm}^a \partial_a - i s e^{\pm i \varphi} \csc \theta \\ &= e^{\pm i \varphi} [\pm i \partial_\theta - \cot \theta \partial_\varphi - i s \csc \theta], \end{aligned} \tag{20}$$

with  $s$  the spin weight of the function on which the operator is acting:  $s = -2$  for  $H$ , and  $s = +2$  for  $\bar{H}$  (see [Appendix A](#)). The last result implies that

$$(\mathfrak{L}_{\xi_{\pm}} h_{ab}) \partial_t h^{ab} = \hat{J}_{\pm} H \partial_t \bar{H} + \hat{J}_{\pm} \bar{H} \partial_t H = 2 \operatorname{Re} \left\{ \hat{J}_{\pm} H \partial_t \bar{H} \right\}, \tag{21}$$

from which we find

$$(\mathfrak{L}_{\xi_x} h_{ab}) \partial_t h^{ab} = 2 \operatorname{Re} \left\{ \hat{J}_x H \partial_t \bar{H} \right\}, \tag{22}$$

$$(\mathfrak{L}_{\xi_y} h_{ab}) \partial_t h^{ab} = 2 \operatorname{Re} \left\{ \hat{J}_y H \partial_t \bar{H} \right\}. \tag{23}$$

Collecting the results, the flux of angular momentum becomes

$$\frac{dJ_i}{dt} = - \lim_{r \rightarrow \infty} \frac{r^2}{16\pi} \operatorname{Re} \oint \hat{J}_i H \partial_t \bar{H} d\Omega, \tag{24}$$

with the angular momentum operators  $\hat{J}_i$  defined as

$$\begin{aligned} \hat{J}_x &= \frac{1}{2} (\hat{J}_+ + \hat{J}_-) \\ &= -\sin \varphi \partial_\theta - \cos \varphi (\cot \theta \partial_\varphi - i s \csc \theta), \\ \hat{J}_y &= -\frac{i}{2} (\hat{J}_+ - \hat{J}_-) \\ &= +\cos \varphi \partial_\theta - \sin \varphi (\cot \theta \partial_\varphi - i s \csc \theta), \\ \hat{J}_z &= \partial_\varphi. \end{aligned} \tag{25}$$

Notice that, except for a factor of  $-i\hbar$ , these are just the quantum mechanical angular momentum operators with the correct spin weight [18].

We now have expressions for the radiated energy, linear momentum, and angular momentum (more details on the derivation of these expressions can be found in [19]). However, it turns out that the TT coefficients  $h^+$  and  $h^\times$  are not trivial to obtain from a numerical simulation. One of the reasons for this is that during a numerical simulation one obtains the full spacetime metric, and not the background spacetime plus a separate perturbation. One therefore needs to relate these metric perturbations to some geometric quantities, preferably scalars, that can be obtained directly from the available data. In the following Section, we will concentrate on one such quantity that turns out to be ideal for extracting gravitational wave information, namely the Weyl scalar  $\Psi_4$ .

### 3 The Weyl scalar $\Psi_4$

#### 3.1 Definition of $\Psi_4$

It is well known that the scalars needed to completely characterize a given spacetime can be expressed as projections onto a null tetrad of the Weyl tensor. These scalars, known as the *Weyl scalars*, have several interesting properties. For instance, for a suitable chosen tetrad, they can be related directly to the gravitational waves at null infinity. We will be interested mainly in the Weyl scalar  $\Psi_4$ , which is associated with outgoing gravitational radiation and is defined as

$$\Psi_4 := C_{\alpha\beta\mu\nu} k^\alpha \bar{m}^\beta k^\mu \bar{m}^\nu, \tag{26}$$

with  $C_{\alpha\beta\mu\nu}$  the Weyl tensor and where  $k^\mu$  and  $\bar{m}^\mu$  are two vectors of a null tetrad constructed from the orthonormal spherical basis in the following way

$$\begin{aligned}
l^\mu &:= \frac{1}{\sqrt{2}} (\hat{e}_t^\mu + \hat{e}_r^\mu), \\
k^\mu &:= \frac{1}{\sqrt{2}} (\hat{e}_t^\mu - \hat{e}_r^\mu), \\
m^\mu &:= \frac{1}{\sqrt{2}} (\hat{e}_\theta^\mu + i\hat{e}_\phi^\mu), \\
\bar{m}^\mu &:= \frac{1}{\sqrt{2}} (\hat{e}_\theta^\mu - i\hat{e}_\phi^\mu),
\end{aligned} \tag{27}$$

where the vectors  $\hat{e}_t^\mu$ ,  $\hat{e}_r^\mu$ ,  $\hat{e}_\theta^\mu$ , and  $\hat{e}_\phi^\mu$  are the usual orthonormal basis induced by the spherical coordinates. The Weyl scalars ( $\Psi_0$ ,  $\Psi_1$ ,  $\Psi_2$ ,  $\Psi_3$ ) are similarly defined as different contractions of the Weyl tensor with the null tetrad.

It turns out that the complex quantity  $H = h^+ - ih^\times$ , defined in the previous Section, can in fact also be written in terms of the Weyl scalar  $\Psi_4$ . In order to see this notice first that if we are in vacuum far from the source of the gravitational waves the Weyl and Riemann tensors coincide (the Ricci tensor vanishes). Using now the standard expression for the Riemann tensor in the linearized approximation one can easily show that, for outgoing plane waves in the TT gauge traveling along the  $r$  direction,  $\Psi_4$  takes the simple form

$$\Psi_4 = -(\ddot{h}^+ - i\ddot{h}^\times) = -\ddot{H}, \tag{28}$$

while all the other Weyl scalars vanish. This implies that for outgoing gravitational waves we can write

$$H = -\int_{-\infty}^t \int_{-\infty}^{t'} \Psi_4 dt'' dt'. \tag{29}$$

### 3.2 Radiated energy and momentum

We can now use Eq. (29) to rewrite the expressions for the radiated energy and momentum derived in the previous Section directly in term of  $\Psi_4$ .<sup>1</sup> However, before doing this it is convenient to project  $\Psi_4$  onto the sphere and describe its angular dependence in terms of the spin-weighted spherical harmonics  ${}_sY^{l,m}$  (see Appendix A). One can easily show that  $\Psi_4$  has spin weight  $s = -2$ , so that we can expand it as

$$\Psi_4 = \sum_{l=2}^{\infty} \sum_{m=-l}^l A^{l,m} \left( {}_{-2}Y^{l,m}(\theta, \phi) \right), \tag{30}$$

<sup>1</sup> The expressions for the radiated energy, linear momentum and  $z$  component of the angular momentum in terms of  $\Psi_4$  (without the multipole expansion) can be found in [13].



with  $A^{l,m}$  the expansion coefficients given by

$$A^{l,m} = \oint \Psi_4 \left( {}_{-2}\bar{Y}^{l,m}(\theta, \phi) \right) d\Omega. \tag{31}$$

Notice that since we are expanding over the harmonics of spin-weight  $s = -2$ , the sum over  $l$  starts at  $l = 2$  (the  ${}_sY^{l,m}$  are only defined for  $l \geq |s|$ ). To simplify the notation, from now on we will drop the limits on the summations and will always assume that the sum over  $l$  starts at 2, while the sum over  $m$  goes from  $-l$  to  $l$ . For summations involving coefficients with indices  $l' = l \pm 1$  and  $m' = m \pm 1$  one should only remember that the corresponding coefficients vanish whenever  $l' < 2$  and  $|m'| > l$ .

In order to calculate the total flux of energy leaving the system we can now use (8) together with the relation between  $H$  and  $\Psi_4$ , Eq. (29), to find

$$\frac{dE}{dt} = \lim_{r \rightarrow \infty} \frac{r^2}{16\pi} \oint \left| \int_{-\infty}^t \Psi_4 dt' \right|^2 d\Omega. \tag{32}$$

Using now the orthogonality of the  ${}_sY^{l,m}$ , we can rewrite the radiated energy as

$$\frac{dE}{dt} = \lim_{r \rightarrow \infty} \frac{r^2}{16\pi} \sum_{l,m} \left| \int_{-\infty}^t A^{l,m} dt' \right|^2. \tag{33}$$

In a similar way, we can express the linear momentum radiated in terms of  $\Psi_4$  and the  $A^{l,m}$  coefficients. Using the expression for the radiated momentum, Eq. (11), and again Eq. (29), we obtain

$$\frac{dP_i}{dt} = \lim_{r \rightarrow \infty} \frac{r^2}{16\pi} \oint l_i \left| \int_{-\infty}^t \Psi_4 dt' \right|^2 d\Omega. \tag{34}$$

Substituting now the multipole expansion of  $\Psi_4$  we find

$$\begin{aligned} \frac{dP_i}{dt} &= \lim_{r \rightarrow \infty} \frac{r^2}{16\pi} \sum_{l,m} \sum_{l',m'} \oint l_i \left( {}_{-2}Y^{l,m} \right) \left( {}_{-2}\bar{Y}^{l',m'} \right) d\Omega \\ &\quad \times \int_{-\infty}^t A^{l,m} dt' \int_{-\infty}^t \bar{A}^{l',m'} dt'. \end{aligned} \tag{35}$$

In order to calculate the integral over the sphere, notice first that the components of the radial unit vector  $l_i$  can be expressed in terms of scalar (i.e., spin zero) spherical harmonics as

$$l_x = \sin \theta \cos \varphi = \sqrt{\frac{2\pi}{3}} \left[ Y^{1,-1} - Y^{1,1} \right], \quad (36)$$

$$l_y = \sin \theta \sin \varphi = i \sqrt{\frac{2\pi}{3}} \left[ Y^{1,-1} + Y^{1,1} \right], \quad (37)$$

$$l_z = \cos \theta = 2 \sqrt{\frac{\pi}{3}} Y^{1,0}. \quad (38)$$

We then see that the flux of linear momentum involves integrals over three spin-weighted spherical harmonics. Such integrals are given in terms of the Wigner  $3\text{-}lm$  symbols with  $l_3 = 1$ , and are also explicitly given in [Appendix A](#).

Instead of  $P_x$  and  $P_y$  it turns out to be easier to work with the complex quantity  $P_+ := P_x + iP_y$ . After a straightforward, but rather long, calculation one finally arrives at the following expressions for the flux of linear momentum

$$\begin{aligned} \frac{dP_+}{dt} &= \lim_{r \rightarrow \infty} \frac{r^2}{8\pi} \sum_{l,m} \int_{-\infty}^t dt' A^{l,m} \\ &\quad \times \int_{-\infty}^t dt' \left( a_{l,m} \bar{A}^{l,m+1} + b_{l,-m} \bar{A}^{l-1,m+1} - b_{l+1,m+1} \bar{A}^{l+1,m+1} \right), \end{aligned} \quad (39)$$

$$\frac{dP_z}{dt} = \lim_{r \rightarrow \infty} \frac{r^2}{16\pi} \sum_{l,m} \int_{-\infty}^t dt' A^{l,m} \int_{-\infty}^t dt' \left( c_{l,m} \bar{A}^{l,m} + d_{l,m} \bar{A}^{l-1,m} + d_{l+1,m} \bar{A}^{l+1,m} \right), \quad (40)$$

where the coefficients  $(a_{l,m}, b_{l,m}, c_{l,m}, d_{l,m})$  are given by

$$a_{l,m} = \frac{\sqrt{(l-m)(l+m+1)}}{l(l+1)}, \quad (41)$$

$$b_{l,m} = \frac{1}{2l} \sqrt{\frac{(l-2)(l+2)(l+m)(l+m-1)}{(2l-1)(2l+1)}}, \quad (42)$$

$$c_{l,m} = \frac{2m}{l(l+1)}, \quad (43)$$

$$d_{l,m} = \frac{1}{l} \sqrt{\frac{(l-2)(l+2)(l-m)(l+m)}{(2l-1)(2l+1)}}. \quad (44)$$

Finally, for the flux of angular momentum we go back to Eq. (24) to obtain

$$\frac{dJ_i}{dt} = - \lim_{r \rightarrow \infty} \frac{r^2}{16\pi} \operatorname{Re} \left\{ \oint \left( \int_{-\infty}^t \bar{\Psi}_4 dt' \right) \times \hat{J}_i \left( \int_{-\infty}^t \int_{-\infty}^{t'} \Psi_4 dt'' dt' \right) d\Omega \right\}. \quad (45)$$

Expressing  $\Psi_4$  in terms of its multipole expansion and integrating over the sphere we now find

$$\begin{aligned} \frac{dJ_i}{dt} = & - \lim_{r \rightarrow \infty} \frac{r^2}{16\pi} \operatorname{Re} \left\{ \sum_{l,m} \sum_{l',m'} \int_{-\infty}^t \bar{A}^{l',m'} dt' \right. \\ & \left. \times \int_{-\infty}^t \int_{-\infty}^{t'} A^{l,m} dt'' dt' \times \oint -_2 \bar{Y}^{l',m'} \hat{J}_i \left( -_2 Y^{l,m} \right) d\Omega \right\}, \end{aligned} \tag{46}$$

where the action of the angular momentum operators  $\hat{J}_i$  on the spin-weighted spherical harmonics is given in [Appendix A](#). We again obtain integrals that involve products of two spin-weighted spherical harmonics which satisfy the usual orthonormalization relations. One can then easily find the following expressions for the angular momentum carried by the gravitational waves

$$\begin{aligned} \frac{dJ_x}{dt} = & - \lim_{r \rightarrow \infty} \frac{ir^2}{32\pi} \operatorname{Im} \left\{ \sum_{l,m} \int_{-\infty}^t \int_{-\infty}^{t'} A^{l,m} dt'' dt' \right. \\ & \left. \times \int_{-\infty}^t \left( f_{l,m} \bar{A}^{l,m+1} + f_{l,-m} \bar{A}^{l,m-1} \right) dt' \right\}, \end{aligned} \tag{47}$$

$$\begin{aligned} \frac{dJ_y}{dt} = & - \lim_{r \rightarrow \infty} \frac{r^2}{32\pi} \operatorname{Re} \left\{ \sum_{l,m} \int_{-\infty}^t \int_{-\infty}^{t'} A^{l,m} dt'' dt' \right. \\ & \left. \times \int_{-\infty}^t \left( f_{l,m} \bar{A}^{l,m+1} - f_{l,-m} \bar{A}^{l,m-1} \right) dt' \right\}, \end{aligned} \tag{48}$$

$$\frac{dJ_z}{dt} = - \lim_{r \rightarrow \infty} \frac{ir^2}{16\pi} \operatorname{Im} \left\{ \sum_{l,m} m \int_{-\infty}^t \int_{-\infty}^{t'} A^{l,m} dt' dt'' \int_{-\infty}^t \bar{A}^{l,m} dt' \right\}, \tag{49}$$

with

$$\begin{aligned} f_{l,m} & := \sqrt{(l-m)(l+m+1)} \\ & = \sqrt{l(l+1) - m(m+1)}, \end{aligned} \tag{50}$$

and where we use the convention that  $\operatorname{Im}(a + ib) = ib$ , for  $a$  and  $b$  real. These last expressions have also been recently derived following a different route by Lousto and Zlochower in [\[20\]](#). The different factor of  $1/4$  between our expression and the expressions of [\[20\]](#) is due to a different normalization of the null tetrad used to define  $\Psi_4$ .

## 4 Black hole perturbation theory

In order to relate the expressions for the radiated energy and momentum in terms of the Weyl scalar  $\Psi_4$  to the standard ones in terms of gauge invariant perturbations, we will present here a brief discussion of some of the ideas behind the theory of perturbations of a Schwarzschild black hole (a more detailed discussion can be seen in, e.g. [21–23]).

### 4.1 Multipole expansion

Consider a metric of the form (1), with the background metric  $\hat{g}_{\mu\nu}$  given by the Schwarzschild metric in standard coordinates:

$$\hat{g}_{\mu\nu} dx^\mu dx^\nu = f(r) dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\Omega^2, \quad (51)$$

with  $f(r) = (1 - 2M/r)$ . Because of the spherical symmetry of the background, it is convenient to think of the full spacetime as the product of a Lorentzian two-dimensional manifold  $M^2$  associated to the coordinates  $(t, r)$ , and the two-sphere of unit radius  $S^2$  associated to  $(\theta, \varphi)$ :

$$ds^2 = g_{AB} dx^A dx^B + r^2 \Omega_{ab} dx^a dx^b, \quad (52)$$

where  $\Omega_{ab}$  is the metric on  $S^2$ :  $\Omega_{ab} = \text{diag}(1, \sin^2 \theta)$ . Here, and in what follows, we will use upper case indices  $(A, B, \dots)$  to represent the coordinates in  $M^2$ , and lower case indices  $(a, b, \dots)$  for coordinates in  $S^2$ . We will also distinguish covariant derivatives in the full spacetime from covariant derivatives in the submanifolds:  $\nabla_\mu$  will represent covariant derivatives in spacetime, while  $D_A$  and  $D_a$  will denote covariant derivatives in  $M^2$  and  $S^2$ , respectively.

We will now consider an expansion of the metric perturbation  $h_{\mu\nu}$  in multipoles using spherical harmonics  $Y^{l,m}(\theta, \varphi)$ . Such a decomposition naturally separates the perturbation into even (or axial) modes and odd (or polar) modes: even modes are those that transform as  $(-1)^l$  under a parity transformation  $(\theta, \varphi) \rightarrow (\pi - \theta, \pi + \varphi)$ , while odd modes transform instead as  $(-1)^{l+1}$ .

In order to decompose  $h_{\mu\nu}$  one further needs to introduce the scalar, vector and tensor harmonics (these tensorial properties refer only to transformations in the unit sphere). Scalar harmonics are the usual functions  $Y^{l,m}$ . Vector harmonics, on the other hand, come in two different types. The even vector harmonics are simply defined as the gradient of the scalar harmonics on the sphere

$$Y_a^{l,m} := D_a Y^{l,m}, \quad (53)$$

while the odd vector harmonics are

$$X_a^{l,m} := -\epsilon_a^b D_b Y^{l,m} = -\epsilon_{ac} \Omega^{cb} D_b Y^{l,m}, \quad (54)$$

where  $\epsilon_{ab}$  is the Levi-Civita tensor on the two-sphere ( $\epsilon_{\theta\varphi} = -\epsilon_{\varphi\theta} = \Omega^{1/2} = \sin \theta$ ).

Similarly, one can define tensor harmonics of even and odd type. Even tensor harmonics can be constructed in two ways, either by multiplying the scalar harmonics with the angular metric  $\Omega_{ab}$ , or by taking a second covariant derivative of the  $Y^{l,m}$ . However, it turns out that these functions do not form a linearly independent set. Instead of the  $D_a D_b Y^{l,m}$  it is better to use the so-called ‘‘Zerilli–Mathews’’ tensor harmonics (see, e.g. [21]) defined as

$$Z_{ab}^{l,m} := D_a D_b Y^{l,m} + \frac{1}{2} l(l+1) \Omega_{ab} Y^{l,m}. \tag{55}$$

It is easy to show that this tensor is traceless, which implies that the resultant functions are linearly independent. Odd parity tensor harmonics, on the other hand, can be constructed in only one way, namely

$$X_{ab}^{l,m} = \frac{1}{2} (D_a X_b^{l,m} + D_b X_a^{l,m}). \tag{56}$$

Having defined the vector and tensor harmonics, the perturbed metric is expanded in multipoles, and separated into its even sector given by

$$\left( h_{AB}^{l,m} \right)_{\text{even}} = H_{AB}^{l,m} Y^{l,m}, \tag{57}$$

$$\left( h_{Ab}^{l,m} \right)_{\text{even}} = H_A^{l,m} Y_b^{l,m}, \tag{58}$$

$$\left( h_{ab}^{l,m} \right)_{\text{even}} = r^2 (K^{l,m} \Omega_{ab} Y^{l,m} + G^{l,m} Z_{ab}^{l,m}), \tag{59}$$

notice that for the case  $l = 1$  the tensor  $Z_{ab}^{l,m}$  vanishes, and the odd sector is given by

$$\left( h_{AB}^{l,m} \right)_{\text{odd}} = 0, \tag{60}$$

$$\left( h_{Ab}^{l,m} \right)_{\text{odd}} = h_A^{l,m} X_b^{l,m}, \tag{61}$$

$$\left( h_{ab}^{l,m} \right)_{\text{odd}} = h^{l,m} X_{ab}^{l,m}, \tag{62}$$

where the coefficients  $(H_{AB}^{l,m}, H_A^{l,m}, K^{l,m}, G^{l,m}, h_A^{l,m}, h^{l,m})$  are in general the functions of  $r$  and  $t$ .

Notice that, since  $Y^{00}$  is a constant, both vector and tensor harmonics vanish for  $l = 0$ . On the other hand, for  $l = 1$  the vector harmonics do not vanish, but the tensor harmonics can still be easily shown to vanish from the explicit expressions for  $Y^{1,m}$ . This means that vector harmonics are only non-zero for  $l \geq 1$ , and tensor harmonics for  $l \geq 2$ . The scalar mode with  $l = 0$  can be interpreted as a variation in the mass of the Schwarzschild spacetime, while the even mode with  $l = 1$  is just gauge and can be removed under a suitable transformation [22–24]. On the other hand, the odd mode with  $l = 1$  can be interpreted as an infinitesimal angular momentum contribution, i.e., a ‘‘Kerr’’ mode (a detailed discussion about this point can be found in [23]). We

will therefore only be interested in modes with  $l \geq 2$ , just as it happened with the expansion of  $\Psi_4$ .

### 4.2 Gauge invariant perturbations

As we have seen, even parity perturbations are characterized by the coefficients  $(H_{AB}^{l,m}, H_A^{l,m}, K^{l,m}, G^{l,m})$ . These coefficients are clearly coordinate dependent, and in particular change under infinitesimal coordinate transformations of the form  $x^\mu \rightarrow x^\mu + \xi^\mu$ , with  $|\xi^\mu| \ll 1$ . However, it turns out that one can construct gauge invariant combinations of coefficients. Two such invariant combinations are (for details see, e.g. [21,22,25])

$$\tilde{K}^{l,m} := K^{l,m} + \frac{1}{2} l(l+1) G^{l,m} - \frac{2}{r} r^A \varepsilon_A^{l,m}, \tag{63}$$

$$\tilde{H}_{AB}^{l,m} := H_{AB}^{l,m} - D_A \varepsilon_B^{l,m} - D_B \varepsilon_A^{l,m}, \tag{64}$$

with  $\varepsilon_A^{l,m} := H_A^{l,m} - r^2 D_A G^{l,m} / 2$ . In terms of the gauge invariant perturbations  $\tilde{K}^{l,m}$  and  $\tilde{H}_{AB}^{l,m}$  one defines the so-called ‘‘Zerilli–Moncrief master function’’ as

$$\Psi_{\text{even}}^{l,m} := \frac{2r}{L} \left[ \tilde{K}^{l,m} + \frac{2r^A}{\Lambda} \left( r^B \tilde{H}_{AB}^{l,m} - r D_A \tilde{K}^{l,m} \right) \right], \tag{65}$$

where  $L := l(l+1)$ ,  $\Lambda := (l-1)(l+2) + 6M/r$ , and  $r_A := D_A r$ . This quantity can be shown to obey a simple wavelike equation known as the ‘‘Zerilli equation’’ [11], though we will not go into such details here.

One can also construct a gauge invariant quantity for the case of odd perturbations in the following way (again, see, e.g. [21,22,25])

$$\tilde{h}_A^{l,m} := h_A^{l,m} - \frac{1}{2} r^2 D_A \left( \frac{h^{l,m}}{r^2} \right). \tag{66}$$

In terms of  $\tilde{h}_A^{l,m}$  one now defines the ‘‘Cunningham–Price–Moncrief master function’’ as

$$\begin{aligned} \Psi_{\text{odd}}^{l,m} &:= \frac{2r \epsilon^{AB}}{(l-1)(l+2)} \left[ D_A \tilde{h}_B^{l,m} - \frac{2r_A}{r} \tilde{h}_B^{l,m} \right] \\ &= \frac{2r \epsilon^{AB}}{(l-1)(l+2)} \left[ D_A h_B^{l,m} - \frac{2r_A}{r} h_B^{l,m} \right]. \end{aligned} \tag{67}$$

The second equality shows that  $\Psi_{\text{odd}}$  in fact depends only on  $h_A$  and not on  $h$  (the contributions from  $h$  cancel when contracted with the  $\epsilon^{AB}$ ), but it is nevertheless gauge invariant. Again, using the perturbed vacuum Einstein field equations one can show that  $\Psi_{\text{odd}}$  obeys a simple wavelike equation known as the ‘‘Regge–Wheeler equation’’ [10].

### 4.3 Gravitational radiation in the TT gauge

The advantage of using the gauge invariant quantities introduced above is that they can be naturally related to the gravitational waves in the TT gauge.

Consider first the even parity perturbations. If asymptotically we approach the TT gauge then we will find that  $h_{AB}$  and  $h_{Ab}$  decay much faster than  $h_{ab}$ . According to the multiple expansion we can then ignore the coefficients  $H_{AB}^{l,m}$  and  $H_A^{l,m}$ , and concentrate only on  $K^{l,m}$  and  $G^{l,m}$ . Considering a local orthonormal basis aligned with the angular directions one then finds that

$$(h^+)_{\text{even}}^{l,m} = \frac{G^{l,m}}{2} \left( Z_{\theta\theta}^{l,m} - \frac{Z_{\varphi\varphi}^{l,m}}{\sin^2 \theta} \right), \tag{68}$$

$$(h^\times)_{\text{even}}^{l,m} = G^{l,m} \left( \frac{Z_{\theta\varphi}^{l,m}}{\sin \theta} \right). \tag{69}$$

On the other hand, from the traceless condition one finds that  $K = 0$ . In that case the Zerilli–Moncrief function (65) simplifies to  $\Psi_{\text{even}}^{l,m} = r G^{l,m}$ , which implies that the contribution from even perturbations to the TT metric functions can be written in terms of  $\Psi_{\text{even}}$  as

$$\begin{aligned} (h^+)_{\text{even}}^{l,m} &= \frac{\Psi_{\text{even}}^{l,m}}{2r} \left( Z_{\theta\theta}^{l,m} - \frac{Z_{\varphi\varphi}^{l,m}}{\sin^2 \theta} \right) \\ &= \frac{\Psi_{\text{even}}^{l,m}}{r} \left[ \frac{\partial^2}{\partial \theta^2} + \frac{1}{2} l(l+1) \right] Y^{l,m}, \end{aligned} \tag{70}$$

$$\begin{aligned} (h^\times)_{\text{even}}^{l,m} &= \frac{\Psi_{\text{even}}^{l,m}}{r} \left( \frac{Z_{\theta\varphi}^{l,m}}{\sin \theta} \right) \\ &= \frac{\Psi_{\text{even}}^{l,m}}{r} \left( \frac{i m}{\sin \theta} \right) \left[ \frac{\partial}{\partial \theta} - \cot \theta \right] Y^{l,m}, \end{aligned} \tag{71}$$

where we used the fact that  $\partial_\varphi Y^{l,m} = i m Y^{l,m}$ .

Consider now the odd perturbations. We find,

$$(h^+)_{\text{odd}}^{l,m} = \frac{h^{l,m}}{2r^2} \left( X_{\theta\theta}^{l,m} - \frac{X_{\varphi\varphi}^{l,m}}{\sin^2 \theta} \right), \tag{72}$$

$$(h^\times)_{\text{odd}}^{l,m} = \frac{h^{l,m}}{r^2} \left( \frac{X_{\theta\varphi}^{l,m}}{\sin \theta} \right). \tag{73}$$

We now need to relate  $h^{l,m}$  to  $\Psi_{\text{odd}}^{l,m}$ . In this case, however, we cannot just ignore  $h_A^{l,m}$  in favor of  $h^{l,m}$ , since from the definition of  $\Psi_{\text{odd}}^{l,m}$  we see that it in fact depends only on  $h_A^{l,m}$  and not on  $h^{l,m}$ . Nevertheless, in the TT gauge these quantities are related to

each other. In order to see this, consider the transverse condition on  $h_{\mu a}$ ,  $\nabla^\mu h_{\mu a} = 0$ . Using the multipole expansion, and calculating explicitly the divergence of  $X_{ab}^{l,m}$ , one finds that this condition implies

$$D^A \left( r^2 h_A^{l,m} \right) = \frac{1}{2} (l - 1)(l + 2) h^{l,m}. \tag{74}$$

Remembering now that in the TT gauge one has the extra freedom of taking  $h_{\mu\nu}$  to be purely spatial, plus the fact that asymptotically the metric  $g_{AB}$  is just the Minkowski metric, the previous expression reduces to

$$\partial_r \left( r^2 h_r^{l,m} \right) = \frac{1}{2} (l - 1)(l + 2) h^{l,m}. \tag{75}$$

In the same limit one can also rewrite expression (67) for  $\Psi_{\text{odd}}^{l,m}$  as

$$\Psi_{\text{odd}}^{l,m} = \frac{2r}{(l - 1)(l + 2)} \partial_t h_r^{l,m}. \tag{76}$$

Collecting results we find that

$$\partial_r \left( r \Psi_{\text{odd}}^{l,m} \right) = \partial_t h^{l,m}. \tag{77}$$

Since for an outgoing wave we have  $\partial_t h_r^{l,m} \sim -\partial_r h_r^{l,m}$ , we can integrate the above expression to find

$$h^{l,m} \sim -r \Psi_{\text{odd}}^{l,m}. \tag{78}$$

We can then rewrite the odd metric perturbations as

$$\begin{aligned} (h^+)_{\text{odd}}^{l,m} &= -\frac{\Psi_{\text{odd}}^{l,m}}{2r} \left( X_{\theta\theta}^{l,m} - \frac{X_{\varphi\varphi}^{l,m}}{\sin^2 \theta} \right) \\ &= -\frac{\Psi_{\text{odd}}^{l,m}}{r} \left( \frac{im}{\sin \theta} \right) \left[ \frac{\partial}{\partial \theta} - \cot \theta \right] Y^{l,m}, \end{aligned} \tag{79}$$

$$\begin{aligned} (h^\times)_{\text{odd}}^{l,m} &= -\frac{\Psi_{\text{odd}}^{l,m}}{r} \left( \frac{X_{\theta\varphi}^{l,m}}{\sin \theta} \right) \\ &= \frac{\Psi_{\text{odd}}^{l,m}}{r} \left[ \frac{\partial^2}{\partial \theta^2} + \frac{1}{2} l(l + 1) \right] Y^{l,m}. \end{aligned} \tag{80}$$



The full TT coefficients  $h^+$  and  $h^\times$  then take the form

$$h^+ = \frac{1}{2r} \sum_{l,m} \left[ \Psi_{\text{even}}^{l,m} \left( Z_{\theta\theta}^{l,m} - \frac{Z_{\varphi\varphi}^{l,m}}{\sin^2\theta} \right) - \Psi_{\text{odd}}^{l,m} \left( X_{\theta\theta}^{l,m} - \frac{X_{\varphi\varphi}^{l,m}}{\sin^2\theta} \right) \right], \tag{81}$$

$$h^\times = \frac{1}{r} \sum_{l,m} \left[ \Psi_{\text{even}}^{l,m} \left( \frac{Z_{\theta\varphi}^{l,m}}{\sin\theta} \right) - \Psi_{\text{odd}}^{l,m} \left( \frac{X_{\theta\varphi}^{l,m}}{\sin\theta} \right) \right]. \tag{82}$$

Notice that while the functions  $(\Psi_{\text{even}}^{l,m}, \Psi_{\text{odd}}^{l,m})$  are in general complex, the TT coefficients  $h^+$  and  $h^\times$  must be real. Using the properties of the spherical harmonics under complex conjugation one can easily show that this implies

$$\bar{\Psi}_{\text{even}}^{l,m} = (-1)^m \Psi_{\text{even}}^{l,-m}, \quad \bar{\Psi}_{\text{odd}}^{l,m} = (-1)^m \Psi_{\text{odd}}^{l,-m}. \tag{83}$$

One can also rewrite the expressions above in terms of spin-weighted spherical harmonics. In order to do this it is in fact more convenient to consider the complex combination  $H$ , for which we find

$$H = \frac{1}{2r} \sum_{l,m} \sqrt{\frac{(l+2)!}{(l-2)!}} (\Psi_{\text{even}} + i \Psi_{\text{odd}}) {}_{-2}Y^{l,m}. \tag{84}$$

At this point it is important to mention one very common convention used in numerical relativity. We start by considering a different odd master function originally introduced by Moncrief [12]:

$$\begin{aligned} Q_M^{l,m} &:= \frac{2r^A \tilde{h}_A^{l,m}}{r} \\ &= \frac{2r^A}{r} \left( h_A^{l,m} - \frac{1}{2} D_A h^{l,m} + \frac{r_A}{r} h^{l,m} \right). \end{aligned} \tag{85}$$

Notice that with this definition  $Q_M^{l,m}$  is clearly scalar and gauge invariant. The Moncrief function just defined has traditionally been the most common choice to study the odd perturbations of Schwarzschild, and because of this many numerical implementations use this function instead of  $\Psi_{\text{odd}}^{l,m}$ . It is possible to show that asymptotically, and in the TT gauge,  $Q_M^{l,m}$  reduces to

$$Q_M^{l,m} \sim -\partial_t \Psi_{\text{odd}}^{l,m}. \tag{86}$$

One finally introduces the following rescaling

$$Q_{\text{even}}^{l,m} := \sqrt{\frac{(l+2)!}{2(l-2)!}} \Psi_{\text{even}}^{l,m}, \tag{87}$$

$$Q_{\text{odd}}^{l,m} := \sqrt{\frac{(l+2)!}{2(l-2)!}} Q_M^{l,m}. \tag{88}$$

In terms of  $Q_{\text{even}}^{l,m}$  and  $Q_{\text{odd}}^{l,m}$ , one now finds for the complex quantity  $H$ :

$$H = \frac{1}{\sqrt{2}r} \sum_{l,m} \left[ Q_{\text{even}}^{l,m} - i \int_{-\infty}^t Q_{\text{odd}}^{l,m} dt' \right] {}_{-2}Y^{l,m}. \tag{89}$$

#### 4.4 Radiated energy and momentum

Just as we did for the case of  $\Psi_4$ , we can now express the radiated energy, linear momentum and angular momentum in terms of the master functions  $(\Psi_{\text{even}}^{l,m}, \Psi_{\text{odd}}^{l,m})$  and/or  $(Q_{\text{even}}^{l,m}, Q_{\text{odd}}^{l,m})$ . There are two ways in which one can do these calculations. One approach is to substitute directly equations (84) and (89) into the expressions for the radiated energy and momentum. It is much easier, however, to start from the expressions for the radiated energy and momentum in terms of the multipole expansion of  $\Psi_4$ , and relate the coefficients  $A^{l,m}$  to  $(\Psi_{\text{even}}^{l,m}, \Psi_{\text{odd}}^{l,m})$  and  $(Q_{\text{even}}^{l,m}, Q_{\text{odd}}^{l,m})$ .

Comparing the multipole expansion for  $\Psi_4$  with the expansions for the metric perturbations, and using the fact asymptotically  $\Psi_4 = -\ddot{H}$ , we find

$$\begin{aligned} A^{l,m} &= -\frac{1}{2r} \sqrt{\frac{(l+2)!}{(l-2)!}} \left( \ddot{\Psi}_{\text{even}}^{l,m} + i \ddot{\Psi}_{\text{odd}}^{l,m} \right) \\ &= -\frac{1}{\sqrt{2}r} \left( \ddot{Q}_{\text{even}}^{l,m} - i \dot{Q}_{\text{odd}}^{l,m} \right). \end{aligned} \tag{90}$$

Using this, one can translate expressions in terms of the  $A^{l,m}$  directly into expressions in terms of  $(\Psi_{\text{even}}^{l,m}, \Psi_{\text{odd}}^{l,m})$  and/or  $(Q_{\text{even}}^{l,m}, Q_{\text{odd}}^{l,m})$ .

Let us consider first the radiated energy. Using equations (33) and (90), one immediately obtains

$$\begin{aligned} \frac{dE}{dt} &= \frac{1}{64\pi} \sum_{l,m} \frac{(l+2)!}{(l-2)!} \left( \left| \dot{\Psi}_{\text{even}}^{l,m} \right|^2 + \left| \dot{\Psi}_{\text{odd}}^{l,m} \right|^2 \right) \\ &= \frac{1}{32\pi} \sum_{l,m} \left( \left| \dot{Q}_{\text{even}}^{l,m} \right|^2 + \left| \dot{Q}_{\text{odd}}^{l,m} \right|^2 \right), \end{aligned} \tag{91}$$

where, in order to derive these expressions one must use the fact that, as a consequence of (83),

$$\sum_m \left( \dot{\Psi}_{\text{even}}^{l,m} \dot{\Psi}_{\text{odd}}^{l,m} - \dot{\Psi}_{\text{odd}}^{l,m} \dot{\Psi}_{\text{even}}^{l,m} \right) = 0. \tag{92}$$

For the linear momentum we start from Eqs. (39)–(40), and again use (90). The calculation is now considerably longer, and in order to simplify the expressions one must make use several times of (83) (or its equivalent in terms of the  $Q$ 's). The final

result is

$$\frac{dP_+}{dt} = \frac{1}{8\pi} \sum_{l,m} \left[ i a_{l,m} \dot{Q}_{\text{even}}^{l,m} \bar{Q}_{\text{odd}}^{l,m+1} - b_{l+1,m+1} \left( \dot{Q}_{\text{even}}^{l,m} \dot{Q}_{\text{even}}^{l+1,m+1} + Q_{\text{odd}}^{l,m} \bar{Q}_{\text{odd}}^{l+1,m+1} \right) \right], \tag{93}$$

$$\frac{dP_z}{dt} = \frac{1}{16\pi} \sum_{l,m} \left[ i c_{l,m} \dot{Q}_{\text{even}}^{l,m} \bar{Q}_{\text{odd}}^{l,m} + d_{l+1,m} \left( \dot{Q}_{\text{even}}^{l,m} \dot{Q}_{\text{even}}^{l+1,m} + Q_{\text{odd}}^{l,m} \bar{Q}_{\text{odd}}^{l+1,m} \right) \right], \tag{94}$$

with the coefficients  $(a_{l,m}, b_{l,m}, c_{l,m}, d_{l,m})$  the same as before. The last expressions are written only in terms of  $(Q_{\text{even}}^{l,m}, Q_{\text{odd}}^{l,m})$ , but it is trivial to rewrite them in terms of  $(\Psi_{\text{even}}^{l,m}, \Psi_{\text{odd}}^{l,m})$ . These expressions can be easily shown to be equivalent to those recently derived by Pollney et al. in [9], and by Sopuerta et al. in [16], (but one must be careful when comparing with the first reference, as their sums over  $m$  go only from 0 to  $l$ ).

In a similar way we can obtain expressions for the radiated angular momentum starting from Eqs. (47)–(49). One now finds, using some algebraic equations that,

$$\frac{dJ_x}{dt} = \frac{i}{32\pi} \text{Im} \sum_{l,m} f_{l,m} \left( \bar{Q}_{\text{even}}^{l,m} \dot{Q}_{\text{even}}^{l,m+1} + \bar{P}_{\text{odd}}^{l,m} Q_{\text{odd}}^{l,m+1} \right), \tag{95}$$

$$\frac{dJ_y}{dt} = -\frac{1}{32\pi} \text{Re} \sum_{l,m} f_{l,m} \left( \bar{Q}_{\text{even}}^{l,m} \dot{Q}_{\text{even}}^{l,m+1} + \bar{P}_{\text{odd}}^{l,m} Q_{\text{odd}}^{l,m+1} \right), \tag{96}$$

$$\frac{dJ_z}{dt} = \frac{i}{32\pi} \sum_{l,m} m \left( \dot{Q}_{\text{even}}^{l,m} \bar{Q}_{\text{even}}^{l,m} + Q_{\text{odd}}^{l,m} \bar{P}_{\text{odd}}^{l,m} \right), \tag{97}$$

where we have defined

$$P_{\text{odd}}^{l,m} := \int_{-\infty}^t Q_{\text{odd}}^{l,m} dt', \tag{98}$$

and where again the coefficients  $f_{l,m}$  are the same as before. Notice that the expressions for  $dJ_x/dt$  and  $dJ_y/dt$  are manifestly real. For  $dJ_z/dt$  the term inside the sum can be easily shown to be purely imaginary, so that the final result is also real.

Equivalent expressions to the set of Eqs. (33), (39)–(40) and (47)–(49) for the energy and momenta carried away by gravitational waves in terms of  $\Psi_4$ , or the set (91), (93)–(94) and (95)–(97) in terms of gauge invariant perturbations, were derived by Thorne in [15]. One can directly relate them to the results presented here by noticing

that

$$\begin{aligned} A^{l,m} &= \frac{1}{\sqrt{2}r} \left[ {}^{(l+2)}I^{l,m} - i \left( {}^{(l+2)}S^{l,m} \right) \right], \\ \bar{A}^{l,m} &= \frac{(-1)^m}{\sqrt{2}r} \left[ {}^{(l+2)}I^{l,-m} + i \left( {}^{(l+2)}S^{l,-m} \right) \right], \end{aligned} \quad (99)$$

where in Thorne's notation  $I^{l,m}(t-r)$  is the *mass multipole momenta* of the radiation field,  $S^{l,m}(t-r)$  is the *current multipole momenta*, and where  ${}^{(l)}I^{l,m}$  and  ${}^{(l)}S^{l,m}$  denote the  $l$ th time derivative of these quantities.

As a final comment, notice that in order to simplify the notation in this Section we have not explicitly introduced the limit of infinite radius. Nevertheless, this limit should be understood since all the results are valid only in the weak field approximation.

## 5 Conclusions

In this paper we have considered explicit expressions for the energy, linear momentum and angular momentum radiated by an isolated system in the form of gravitational waves. Starting from a small perturbation  $h_{\mu\nu}$  a background metric  $\hat{g}_{\mu\nu}$ , and working in the transverse-traceless gauge, we have reviewed the standard expressions for the radiated energy and momentum based on the Isaacson stress-energy tensor. Introducing the Weyl scalar  $\Psi_4$  and its multipole expansion in terms of spin-weighted spherical harmonics, we have computed explicit expressions for the radiated energy and momentum in terms of the expansion coefficients  $A^{l,m}$ . Finally, we have also presented multipole expansions in terms of the gauge invariant perturbations of a Schwarzschild spacetime. In particular, the expressions in terms of the expansion of the Weyl scalar  $\Psi_4$  have the advantage of avoiding the need for separating out a Schwarzschild "background" in standard coordinates from a numerically generated spacetime that can be in an arbitrary gauge. Nevertheless, one still needs to find a suitable tetrad to calculate the scalar  $\Psi_4$ , and as yet there is no standard procedure to find it (though some recent progress has been made on this issue, see, e.g. [26]). However, as long as one chooses a tetrad that approaches the standard outgoing null tetrad in flat space for large  $r$  the asymptotic expressions will be correct.

Although most of the expressions derived here are known, as far as we know they have never appeared together in the literature. We have taken great care to explain the origin of all relevant expressions, using clear and consistent conventions and definitions. We believe that having all these expressions together will be extremely useful when calculating radiated energy and momenta from numerical simulations of astrophysical systems. As a final comment, we have at our disposal a Mathematica script to find the energy and momenta carried by gravitational waves, and would be happy to provide it to interested readers upon request.

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### Appendix A: Spin-weighted spherical harmonics

In this appendix we will discuss some important properties of the spin-weighted spherical harmonics. Spin-weighted spherical harmonics were first introduced by Newman and Penrose [27] for the study of gravitational radiation, but they can also be used to study solutions of the Maxwell equations, the Dirac equation, or in fact dynamical equations for fields of arbitrary spin.

Consider a complex function  $f$  on the sphere that might correspond to some combination of components of a tensorial (or spinorial) object in the orthonormal basis  $(\hat{e}_r, \hat{e}_\theta, \hat{e}_\varphi)$  induced by the spherical coordinates  $(r, \theta, \varphi)$ . We will say that  $f$  has spin weight  $s$  if, under a rotation of the angular basis  $(\hat{e}_\theta, \hat{e}_\varphi)$  by an angle  $\psi$ , it transforms as  $f \rightarrow e^{-is\psi} f$ . A trivial example is a scalar function whose spin weight is clearly zero. A more interesting example corresponds to a three-dimensional vector  $\mathbf{v}$  with components  $(v^{\hat{r}}, v^{\hat{\theta}}, v^{\hat{\phi}})$ . Notice that these components are different from those in the coordinate basis (which is not orthonormal), and are related to them through  $(v^{\hat{r}}, v^{\hat{\theta}}, v^{\hat{\phi}}) = (v^r, r v^\theta, r \sin \theta v^\phi)$ . Define now two unit complex vectors as

$$\hat{e}_\pm := (\hat{e}_\theta \mp i \hat{e}_\varphi) / \sqrt{2}. \tag{100}$$

The vector  $\mathbf{v}$  can then be written as

$$\mathbf{v} = v^0 \hat{e}_r + v^+ \hat{e}_+ + v^- \hat{e}_-, \tag{101}$$

where  $v^0 := v^{\hat{r}}, v^\pm := (v^{\hat{\theta}} \pm i v^{\hat{\phi}}) / \sqrt{2}$ . By considering a rotation of the vectors  $(\hat{e}_\theta, \hat{e}_\varphi)$  by an angle  $\psi$  it is now easy to see that  $v^0$  has spin weight zero, while the spin weight of  $v^\pm$  is  $\pm 1$ .

The spin-weighted spherical harmonics, denoted by  ${}_s \bar{Y}^{l,m}(\theta, \varphi)$ , form a basis for the space of functions with definite spin weight  $s$ . They can be introduced in a number of different ways. One can start by defining the operators

$$\begin{aligned} \bar{\partial} f &:= -\sin^s \theta \left( \partial_\theta + \frac{i}{\sin \theta} \partial_\varphi \right) (f \sin^{-s} \theta) \\ &= -\left( \partial_\theta + \frac{i}{\sin \theta} \partial_\varphi - s \cot \theta \right) f, \end{aligned} \tag{102}$$

$$\begin{aligned} \bar{\partial} f &:= -\sin^{-s} \theta \left( \partial_\theta - \frac{i}{\sin \theta} \partial_\varphi \right) (f \sin^s \theta) \\ &= -\left( \partial_\theta - \frac{i}{\sin \theta} \partial_\varphi + s \cot \theta \right) f, \end{aligned} \tag{103}$$

where  $s$  is the spin weight of  $f$ . The spin-weighted spherical harmonics are then defined for  $|m| \leq l$  and  $l \geq |s|$  in terms of the standard spherical harmonics as

$${}_s Y^{l,m} := \left[ \frac{(l-s)!}{(l+s)!} \right]^{1/2} \bar{\partial}^s (Y^{l,m}), \quad s \geq 0, \tag{104}$$

$${}_s Y^{l,m} := (-1)^s \left[ \frac{(l+s)!}{(l-s)!} \right]^{1/2} \bar{\partial}^{-s} (Y^{l,m}), \quad s \leq 0. \quad (105)$$

In particular we have  ${}_0 Y^{l,m} = Y^{l,m}$ . The above definition implies that

$$\bar{\partial} ({}_s Y^{l,m}) = +[(l-s)(l+s+1)]^{1/2} {}_{s+1} Y^{l,m}, \quad (106)$$

$$\bar{\partial} ({}_s Y^{l,m}) = -[(l+s)(l-s+1)]^{1/2} {}_{s-1} Y^{l,m}. \quad (107)$$

Because of this,  $\bar{\partial}$  and  $\bar{\partial}$  are known as the *spin raising* and *spin lowering* operators. One also finds that

$$\bar{\partial}\bar{\partial} ({}_s Y^{l,m}) = -[l(l+1) - s(s+1)] {}_s Y^{l,m}, \quad (108)$$

$$\bar{\partial}\bar{\partial} ({}_s Y^{l,m}) = -[l(l+1) - s(s-1)] {}_s Y^{l,m}, \quad (109)$$

so the  ${}_s Y^{l,m}$  are eigenfunctions of the operators  $\bar{\partial}\bar{\partial}$  and  $\bar{\partial}\bar{\partial}$ , which are generalizations of the Laplace operator on the sphere  $L^2$ :

$$L^2 f := \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta f) + \frac{1}{\sin^2 \theta} \partial_\varphi^2 f. \quad (110)$$

For a function with zero spin weight we in fact find that  $L^2 f = \bar{\partial}\bar{\partial} f = \bar{\partial}\bar{\partial} f$ .

One can also find generalizations of the standard angular momentum operators for the case of non-zero spin weight by looking for operators  $\hat{J}_z$  and  $\hat{J}_\pm$  such that (here we are ignoring the factor  $-i\hbar$  that normally appears in quantum mechanics) [18]

$$\hat{J}_z {}_s Y^{l,m} = i m {}_s Y^{l,m}, \quad (111)$$

$$\hat{J}_\pm {}_s Y^{l,m} = i [(l \mp m)(l+1 \pm m)]^{1/2} {}_s Y^{l,m \pm 1}. \quad (112)$$

One then finds that such operators must have the form

$$\hat{J}_z = \partial_\varphi, \quad (113)$$

$$\hat{J}_\pm = e^{\pm i\varphi} [\pm i \partial_\theta - \cot \theta \partial_\varphi - i s \csc \theta]. \quad (114)$$

The operators for the  $x$  and  $y$  components of the angular momentum are then simply obtained from  $\hat{J}_\pm = \hat{J}_x \pm i \hat{J}_y$ , so that we find:

$$\hat{J}_x = (\hat{J}_+ + \hat{J}_-) / 2, \quad \hat{J}_y = -i (\hat{J}_+ - \hat{J}_-) / 2, \quad (115)$$

There are several important properties of the spin-weighted spherical harmonics that can be obtained directly from their definition. In the first place, one can show that the complex conjugate of  ${}_s Y^{l,m}$  is given by

$${}_s \bar{Y}^{l,m}(\theta, \varphi) = (-1)^{s+m} {}_{-s} Y^{l,-m}(\theta, \varphi). \quad (116)$$

Also, the different  ${}_s Y^{l,m}$  are orthonormal:

$$\oint {}_s Y^{l,m}(\theta, \varphi) {}_{s'} \bar{Y}^{l',m'}(\theta, \varphi) d\Omega = \delta_{ss'} \delta_{ll'} \delta_{mm'}, \tag{117}$$

The integral of three spin-weighted spherical harmonics is also frequently needed, for example in the calculation of the momentum flux of gravitational waves, and can be expressed in general as

$$\begin{aligned} &\oint {}_{s_1} Y^{l_1,m_1}(\theta, \varphi) {}_{s_2} Y^{l_2,m_2}(\theta, \varphi) {}_{s_3} Y^{l_3,m_3}(\theta, \varphi) d\Omega \\ &= \left[ \frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi} \right]^{1/2} \begin{pmatrix} l_1 & l_2 & l_3 \\ -s_1 & -s_2 & -s_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}. \end{aligned} \tag{118}$$

The above expression involves the Wigner  $3-lm$  symbols, which are related to the standard Clebsch–Gordan coefficients  $\langle l_1, m_1, l_2, m_2 | j_3, m_3 \rangle$  through

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(-1)^{l_1-l_2-m_3}}{\sqrt{2l_3+1}} \langle l_1, m_1, l_2, m_2 | l_3, -m_3 \rangle. \tag{119}$$

The Clebsch–Gordan coefficients arise from the addition of angular momentum in quantum mechanics, and correspond to the coefficients of the expansion of an eigenstate  $|L, M\rangle$  with total angular momentum  $L$  and projection  $M$ , in terms of a basis formed by the product of the individual eigenstates  $|l_1, m_1\rangle |l_2, m_2\rangle$ . These coefficients have some important symmetries, though these symmetries are easier to express in terms of the  $3-lm$  symbols. In particular, the  $3-lm$  symbols are invariant under an even permutation of columns, and pick up a factor  $(-1)^{l_1+l_2+l_3}$  under an odd permutation. Also, changing the sign of all three  $m$ 's again introduces a factor of  $(-1)^{l_1+l_2+l_3}$ .

A closed expression for the Clebsch–Gordan coefficients was first found by Wigner (see, e.g. [28]). This expression is somewhat simpler when written in terms of the  $3-lm$  symbols and has the form

$$\begin{aligned} &\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{l_1-m_1} \delta_{m_1+m_2,-m_3} \\ &\times \left[ \frac{(l_1+l_2-l_3)! (l_1+l_3-l_2)! (l_2+l_3-l_1)! (l_3+m_3)! (l_3-m_3)!}{(l_1+l_2+l_3+1)! (l_1+m_1)! (l_1-m_1)! (l_2+m_2)! (l_2-m_2)!} \right]^{1/2} \\ &\times \sum_{k \geq 0} \frac{(-1)^k}{k!} \left[ \frac{(l_2+l_3+m_1-k)! (l_1-m_1+k)!}{(l_3-l_1+l_2-k)! (l_3-m_3-k)! (l_1-l_2+m_3+k)!} \right]. \end{aligned} \tag{120}$$

In the above expression the sum runs over all values of  $k$  for which the arguments inside the factorials are non-negative. Also, if the particular combination of  $\{l_i, m_i\}$  is such that the arguments of the factorials outside of the sum are negative, then the

corresponding coefficient vanishes. A more symmetric (though longer) expression that is equivalent to (120) was later derived by Racah [29], but we will not write it here.

In the general case equation (120) is rather complicated, but this is not a serious problem as one can find tables of the most common coefficients in the literature, and even web-based ‘‘Clebsch–Gordan calculators’’. Moreover, in some special cases the coefficients simplify considerably. For example, in the case when  $m_1 = l_1$ ,  $m_2 = l_2$  and  $l_3 = m_3 = l_1 + l_2$  one finds

$$\begin{pmatrix} l_1 & l_2 & l_1 + l_2 \\ l_1 & l_2 & l_1 + l_2 \end{pmatrix} = \frac{1}{\sqrt{2(l_1 + l_2) + 1}}. \quad (121)$$

Another particularly interesting case corresponds to taking  $l_3 = m_3 = 0$  (i.e., zero total angular momentum in quantum mechanics). In that case we find

$$\begin{aligned} \begin{pmatrix} l_1 & l_2 & 0 \\ m_1 & m_2 & 0 \end{pmatrix} &= \langle l_1, m_1, l_2, m_2 | 0, 0 \rangle \\ &= \frac{(-1)^{l_1 - m_1}}{\sqrt{2l_1 + 1}} \delta_{l_1, l_2} \delta_{m_1, -m_2}. \end{aligned} \quad (122)$$

The cases with  $l_3 = 1$  are also interesting as they appear in the expression for the linear momentum carried by gravitational waves. One finds, in particular

$$\begin{aligned} \begin{pmatrix} l_1 & l_2 & 1 \\ m_1 & m_2 & 0 \end{pmatrix} &= (-1)^{l_1 - m_1} \delta_{m_1 + m_2, 0} \left[ \frac{2m_1 \delta_{l_1, l_2}}{\sqrt{(2l_1 + 2)(2l_1 + 1)(2l_1)}} \right. \\ &\quad + \delta_{l_1, l_2 + 1} \sqrt{\frac{(l_1 + m_1)(l_1 - m_1)}{l_1(2l_1 + 1)(2l_1 - 1)}} \\ &\quad \left. - \delta_{l_1 + 1, l_2} \sqrt{\frac{(l_2 - m_2)(l_2 + m_2)}{l_2(2l_2 + 1)(2l_2 - 1)}} \right], \end{aligned} \quad (123)$$

$$\begin{aligned} \begin{pmatrix} l_1 & l_2 & 1 \\ m_1 & m_2 & \pm 1 \end{pmatrix} &= (-1)^{l_1 - m_1} \delta_{m_1 + m_2, \mp 1} \left[ \pm \delta_{l_1, l_2} \sqrt{\frac{(l_1 \mp m_1)(l_1 \mp m_2)}{l_1(2l_1 + 2)(2l_1 + 1)}} \right. \\ &\quad + \delta_{l_1, l_2 + 1} \sqrt{\frac{(l_1 \mp m_1)(l_1 \pm m_2)}{2l_1(2l_1 + 1)(2l_1 - 1)}} \\ &\quad \left. + \delta_{l_1 + 1, l_2} \sqrt{\frac{(l_2 \mp m_2)(l_2 \pm m_1)}{2l_2(2l_2 + 1)(2l_2 - 1)}} \right]. \end{aligned} \quad (124)$$

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